Dynamic Analysis for Free Vibrations of Rotating Sandwich Tapered Beams

C. L. Ko*
Oakland University, Rochester, Michigan

Governing equations for free vibrations of rotating sandwich rectangular and tapered beams are obtained from a previous analysis using the variational method reported by the author. Both Timoshenko's and Euler-Bernoulli's beam models are considered in these formulations. The analysis includes the effects of the variation of the neutral-axis location along the beam due to both the centrifugal effect and a nonsymmetric geometry. This paper presents a pioneer work to determine the vibration characteristics of both symmetrically and nonsymmetrically tapered and rectangular sandwich beams under rotation. Natural frequencies of these beams are determined for various rotating speeds by solving the eigenvalue problems using the finite-difference method. These results indicate that both the asymmetric geometry and the rotating speed can affect the dynamic behavior of a sandwich beam greatly.

Introduction

VIBRATION characteristics of rotating beams can be very important in various engineering applications. Structural components such as helicopter rotor blades, propellers, turbomachinery rotor blades, and robot arms are frequently modeled as rotating cantilever beams. The problem of bending vibration of a rotating beam was first solved by Lo and Renbarger.[1] Boyce et al.[2] later extended the analysis for various boundary conditions and estimated lower and upper bounds on natural frequencies. The influence of a tip mass on the natural frequencies of a uniform rotating cantilever beam was also investigated extensively by Handelman et al.,[3] Boyce and Handelman,[4] Jones,[5] and Hoa.[6] Recently, Tomar and Jain[7,8] determined the thermal effect on frequencies of pretwisted and wedge-shaped beams. Laurensen[9] investigated the influence of mass representation on the equation of motion for rotating structures. Free vibrations of centrifugally stiffened beams were considered by Wright et al.[10] Bruch and Mitchell[11] also modeled a robot link (arm) as a mass-loaded cantilever Timoshenko beam and determined its natural fequencies. However, the rotational action was not considered in their formulations.

Sandwich constructions have been widely applied for various aerospace applications for weight-saving and strength optimization. Bert[12] presented a brief overview for such applications. Vibration problems for rectangular, symmetric, sandwich beams have been investigated extensively for the past three decades. These include the analyses reported by Kobayashi,[13] Krajcinovic,[14] Kimel et al.,[15] Raville et al.,[16] James,[17] Glaser,[18] Clary and Leadbetter,[19] and Bert et al.[20] However, based on the author's knowledge, formulations for bending vibrations for either tapered sandwich beams or rotating sandwich beams have never been reported in literature, except those developed by the author recently.[21] In Ref. 21, the author derived the governing equations for the dynamic response and flexural behavior of general rotating sandwich beams by applying the energy principle and the variational method. However, only results for the semistatic flexural behavior were obtained. The objectives of this investigation, therefore, are to formulate the governing equations and to obtain their solutions for free vibrations of both symmetrically and nonsymmetrically tapered rotating sandwich beams. Dynamic response of a symmetric rectangular sandwich beam is also included in this analysis.

As pointed out in Ref. 21, the neutral axis position of a beam does not necessarily coincide with the centroidal axis if the beam is either under rotation or nonsymmetrically constructed. Analyses which included this consideration have never been reported in literature, except by the author. In addition, the flexural rigidity in the core and the shear deformation of the facings were normally neglected in the dynamic analyses of sandwich beams; therefore, their effects upon the determination of natural frequencies have never been considered, to date.[22] Results obtained in this paper are for the first time calculated from an analysis that included all these considerations.

Descriptions of the Problem

Consider a cantilever sandwich tapered beam, as shown in Fig. 1. The origin of the Cartesian coordinates shown is positioned at the center of the core at the fixed end and the beam is rotating with respect to the y-axis with a constant angular speed Ω. The half-depth of the core is h_1 and the transverse coordinates of the outer edges of the top and the bottom facings are $-h_2$ and h_3, respectively. The Young's modulus, shear modulus, and density of the cord are given as E_1, G_1, and ρ_1, respectively; whereas those for the top and bottom facings are denoted as E_2, G_2, ρ_2 and E_3, G_3, ρ_3, respectively. The length and width of the beam are given as L and b, respectively.

The general governing equations and boundary conditions for the dynamic and flexural behavior were derived by the author.[21] The centrifugal and nonsymmetric effects on the variation of neutral-axis positions were included in the analysis, using both Euler-Bernoulli's and Timoshenko's model. Solutions of these equations consist of two parts: a time-dependent function and a time-independent function. These two functions correspond to solutions for dynamic response and semistatic behavior of the beam. The time-independent solutions for semistatic (or flexural) behavior were presented in Ref. 21. The present paper treats only the case of dynamic response due to free vibrations.

Received Nov. 23, 1987; revision received March 21, 1988.

*Assistant Professor, Department of Mechanical Engineering. Member AIAA.

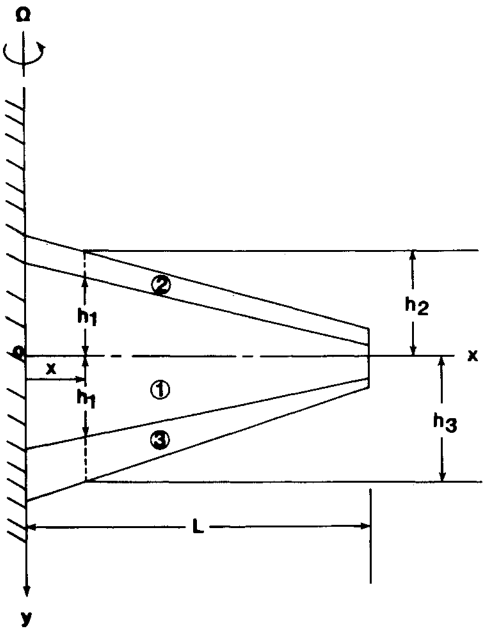

Fig. 1 A rotating sandwich cantilever beam.

Timoshenko Beams

The governing equations for the dynamic response of the beam under free vibration can be deduced from Eqs. (55-57) of Ref. 21, as follows:

$$a_1 \frac{\partial^2 u_o}{\partial t^2} - a_2\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + \frac{\partial}{\partial x}\left[a_5\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) - a_4 \frac{\partial u_o}{\partial x}\right] + \Omega^2\left[a^2\left(\psi + \frac{\partial w_o}{\partial x}\right) - a_1 u_o\right] = 0 \tag{1}$$

$$a_3\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + a_7 \psi - a_2 \frac{\partial^2 u_o}{\partial t^2} + \frac{\partial}{\partial x}\left[a_5 \frac{\partial u_o}{\partial x} - a_6\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right)\right] + \Omega^2\left[a_2 u_o - a_3\left(\psi + \frac{\partial w_o}{\partial x}\right)\right] = 0 \tag{2}$$

$$a_1 \frac{\partial^2 w_o}{\partial t^2} + \frac{\partial}{\partial x}\left[a_2 \frac{\partial^2 u_o}{\partial t^2} - a_3\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right)\right] + \frac{\partial^2}{\partial x^2}\left[a_6\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) - a_5 \frac{\partial u_o}{\partial x}\right] + \Omega^2 \frac{\partial}{\partial x}\left[a_3\left(\psi + \frac{\partial w_o}{\partial x}\right) - a_2 u_o\right] = 0 \tag{3}$$

where u_o, w_o, and ψ are the axial displacement of the core midplane, the transverse displacement, and the angle of rotational deformation of the beam, respectively. The coefficients are defined as

$$a_2 \equiv (2\rho_1 - \rho_2 - \rho_3)h_1 + \rho_2 h_2 + h_3$$

$$a_2 \equiv [\rho_2(h_1^2 - h_2^2) + \rho_3(h_3^2 - h_1^2)]/2$$

$$a_3 \equiv [2\rho_1 h_1^3 + \rho_2(h_2^3 - h_1^3) + \rho_3(h_3^3 - h_1^3)]/3$$

$$a_4 \equiv (2E_1 - E_2 - E_3)h_1 + E_2 h_2 + E_3 h_3$$

$$a_5 \equiv [E_2(h_1^2 - h_2^2) + E_3(h_3^2 - h_1^2)]/2$$

$$a_6 \equiv [2E_1 h_1^3 + E_2(h_2^3 - h_1^3) + E_3(h_3^3 - h_1^3)]/3$$

$$a_7 \equiv 2\kappa_1 G_1 h_1 + \kappa_2 G_2(h_2 - h_1) + \kappa_3 G_3(h_3 - h_1)$$

The quantities κ_1, κ_2, and κ_3 are the shear correction factors for the core and the top and bottom facings, respectively. The boundary conditions of the cantilever beam can be deduced from Eqs. (65), (68), (69), and (70) in Ref. 21, as follows:

At the fixed end ($x = 0$):

$$\psi = u_o = w_o = \frac{\partial w_o}{\partial x} = 0 \tag{4}$$

At the free end ($x = L$):

$$\frac{\partial u_o}{\partial x} = 0 \tag{5}$$

$$\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2} = 0 \tag{6}$$

$$a_2 \frac{\partial^2 u_o}{\partial t^2} - a_3\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + \frac{\partial}{\partial x}\left[a_6\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) - a_5 \frac{\partial u_o}{\partial x}\right] + \Omega^2\left[a_3\left(\psi + \frac{\partial w_o}{\partial x}\right) - a_2 u_o\right] = 0 \tag{7}$$

Euler-Bernoulli Beams

The governing equations for the response of the beam under free vibration can be obtained by simplifying Eqs. (75) and (76) of Ref. 21 as

$$a_1 \frac{\partial^2 u_o}{\partial t^2} - a_2 \frac{\partial^3 w_o}{\partial x \partial t^2} + \frac{\partial}{\partial x}\left(a_5 \frac{\partial^2 w_o}{\partial x^2} - a_4 \frac{\partial u_o}{\partial x}\right) + \Omega^2\left(a_2 \frac{\partial w_o}{\partial x} - a_1 u_o\right) = 0 \tag{8}$$

$$a_1 \frac{\partial^2 w_o}{\partial t^2} + \frac{\partial}{\partial x}\left(a_2 \frac{\partial^2 u_o}{\partial t^2} - a_3 \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + \frac{\partial^2}{\partial x^2}\left(a_6 \frac{\partial^2 w_o}{\partial x^2} - a_5 \frac{\partial u_o}{\partial x}\right) + \Omega^2 \frac{\partial}{\partial x}\left(a_3 \frac{\partial w_o}{\partial x} - a_2 u_o\right) = 0 \tag{9}$$

The boundary conditions reduce to the following:

At the fixed end ($x = 0$):

$$u_o = w_o = \frac{\partial w_o}{\partial x} = 0 \tag{10}$$

At the free end ($x = L$):

$$\frac{\partial u_o}{\partial x} = \frac{\partial^2 w_o}{\partial x^2} = 0 \tag{11}$$

$$a_2 \frac{\partial^2 u_o}{\partial t^2} - a_3 \frac{\partial^3 w_o}{\partial x \partial t^2} + \frac{\partial}{\partial x}\left(a_6 \frac{\partial^2 w_o}{\partial x^2} - a_5 \frac{\partial u_o}{\partial x}\right) + \Omega^2\left(a_3 \frac{\partial w_o}{\partial x} - a_2 u_o\right) = 0 \tag{12}$$

Free Vibration of a Symmetric Rectangular Sandwich Beam

For the case of a rectangular sandwich beam, the coefficients of the governing equations and boundary conditions are constant. In addition, a_2 and a_5 vanish for a beam with symmetric facings; hence, the axial mode of vibration decouples from the transverse mode and the rotational mode. The axial displacement and transverse displacement can be represented by harmonic functions as

$$u_o = U(x)\sin(\omega_1 t + \theta_1) \tag{13}$$

$$w_o = W(x)\sin(\omega_2 t + \theta_2) \tag{14}$$

where ω_1 and ω_2 are natural angular frequencies for the axial and transverse modes, respectively. The quantities θ_1 and θ_2 are the phase angles for the corresponding modes.

Timoshenko Beams

The governing equations, Eqs. (1-3), can be reduced to the following:

$$a_1 \frac{\partial^2 u_o}{\partial t^2} - a_4 \frac{\partial^2 u_o}{\partial x^2} - \Omega^2 a_1 u_o = 0 \tag{15}$$

$$a_3\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + a_7 \psi - a_6\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 w_o}{\partial x^3}\right) - a_3 \Omega^2\left(\psi + \frac{\partial w_o}{\partial x}\right) = 0 \tag{16}$$

$$a_1 \frac{\partial^2 w_o}{\partial t^2} - a_3\left(\frac{\partial^3 \psi}{\partial x \partial t^2} + \frac{\partial^4 w_o}{\partial x^2 \partial t^2}\right) + a_6\left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^4 w_o}{\partial x^4}\right) + a_3 \Omega^2\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) = 0 \tag{17}$$

The boundary condition at $x = L$, Eq. (7), can be simplified to

$$-a_3\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + a_6\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 w_o}{\partial x^3}\right) + \Omega^2 a_3\left(\psi + \frac{\partial w_o}{\partial x}\right) = 0 \tag{18}$$

Assume that the rotational mode ψ has the same frequency and phase angle as the transverse mode w_o. The harmonic expression of this mode becomes

$$\psi = \Phi(x) \sin(\omega_2 t + \theta_2) \tag{19}$$

Substituting Eqs. (13), (14), and (19) into Eqs. (15), (16), and (17), one obtains

$$a_4 \frac{d^2 U}{dx^2} + a_1(\omega_1^2 + \Omega^2) U = 0 \tag{20}$$

$$a_3 \omega_2^2\left(\Phi + \frac{dW}{dx}\right) - a_7 \Phi + a_6\left(\frac{d^2 \Phi}{dx^2} + \frac{d^3 W}{dx^3}\right) + a_3 \Omega^2\left(\Phi + \frac{dW}{dx}\right) = 0 \tag{21}$$

$$a_1 \omega_2^2 W - a_3 \omega_2^2\left(\frac{d\Phi}{dx} + \frac{d^2 W}{dx^2}\right) - a_6\left(\frac{d^3 \Phi}{dx^3} + \frac{d^4 W}{dx^4}\right) - a_3 \Omega^2\left(\frac{d\Phi}{dx} + \frac{d^2 W}{dx^2}\right) = 0 \tag{22}$$

Combining Eqs. (21) and (22), one can obtain

$$\frac{d\Phi}{dx} = \frac{\alpha_1 \omega_2^2}{a_7} W \tag{23}$$

Eq. (22) can then be expressed as follows by utilizing Eq. (23):

$$a_6 a_7 \frac{d^4 W}{dx^4} + \left[a_3 a_7(\omega_2^2 + \Omega^2) + a_1 a_6 \omega_2^2\right] \frac{d^2 W}{dx^2} + \left[a_1 a_3 \omega_2^2(\omega_2^2 + \Omega^2) - a_1 a_7 \omega_2^2\right] W = 0 \tag{24}$$

The boundary conditions can be determined as follows:

At the fixed end ($x = 0$):

$$\Phi = U = W = \frac{dW}{dx} = 0 \tag{25}$$

At the free end ($x = L$):

$$\Phi = \frac{dU}{dx} = 0 \tag{26}$$

$$\frac{d^2 W}{dx^2} + \delta_1 W = 0 \tag{27}$$

$$\frac{d^3 W}{dx^3} + (\delta_1 + \delta_2) \frac{dW}{dx} = 0 \tag{28}$$

where $\delta_1 \equiv a_1 \omega_2^2 / a_7$ and $\delta_2 \equiv a_3(\omega_2^2 + \Omega^2)/a_6$.

The general solution of Eq. (20) that satisfies the boundary conditions at the fixed end can be shown as

$$U = A \sin \alpha x \tag{29}$$

where $\alpha^2 \equiv a_1(\omega_1^2 + \Omega^2)/a_4$ and A is an undetermined constant. The following characteristic equation can be obtained by imposing the boundary conditions at the free end upon the modal shape expressed by Eq. (29):

$$\cos \alpha L = 0 \tag{30}$$

Hence, the natural angular frequencies for the axial modes can be determined by

$$\omega_1^2 = (n - 1/2)^2 \frac{\pi^2 a_4}{L^2 a_1} - \Omega^2, \qquad n = 1,2,3, \ldots \tag{31}$$

The general solution of Eq. (24) can be determined for the following cases:

case 1: $a_3(\omega_2^2 + \Omega^2) > a_7$

$$W = C_1 \cos\beta_1 x + C_2 \sin\beta_1 x + C_3 \cos\beta_2 x + C_4 \sin\beta_2 x \tag{32}$$

case 2: $a_3(\omega_2^2 + \Omega^2) < a_7$

$$W = C_1 \cos\beta_1 x + C_2 \sin\beta_1 x + C_3 \cosh\beta_3 x + C_4 \sinh\beta_3 x \tag{33}$$

case 3: $a_3(\omega_2^2 + \Omega^2) = a_7$

$$W = C_1 \cos\beta_1 x + C_2 \sin\beta_1 x + C_3 + C_4 x \tag{34}$$

where

$$\beta_1 \equiv \sqrt{(\delta_1 + \delta_2 + \delta_3)/2} \qquad \beta_2 \equiv \sqrt{(\delta_1 + \delta_2 - \delta_3)/2}$$

$$\beta_3 \equiv \sqrt{(\delta_3 - \delta_1 - \delta_2)/2} \qquad \delta_3 \equiv \sqrt{(\delta_2 - \delta_1)^2 + 4\delta_4^2}$$

$$\delta_4^2 \equiv a_1 \omega_2^2 / a_6$$

The boundary conditions can be expressed into the following form:

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{35}$$

Expressions for k_{11}, k_{12}, k_{21}, and k_{22} for all three cases are summarized in Appendix A. The natural frequencies for the transverse modes, ω_2, can then be determined from the following characteristic equation:

$$k_{11} k_{22} - k_{12} k_{21} = 0 \tag{36}$$

Euler-Bernoulli Beams

The governing equation, Eq. (8), can be simplified to Eq. (15) and its eigenvalues for axial modes can be determined by using the same relationship for Timoshenko beams. However, Eq. (9) should reduce to

$$a_1\frac{\partial^2 w_o}{\partial t^2} - a_3\frac{\partial^4 w_o}{\partial x^2 \partial t^2} + a_6\frac{\partial^4 w_o}{\partial x^4} + a_3\Omega^2\frac{\partial^2 w_o}{\partial x^2} = 0 \tag{37}$$

The boundary condition expressed in Eq. (12) becomes

$$-a_3\frac{\partial^3 w_o}{\partial x \partial t^2} + a_6\frac{\partial^3 w_o}{\partial x^3} + \Omega^2 a_3\frac{\partial w_o}{\partial x} = 0 \tag{38}$$

Substituting Eqs. (13) and (14) into Eq. (37) yields

$$a_6\frac{d^4 W}{dx^4} + a_3(\omega^2+\Omega^2)\frac{d^2 W}{dx^2} - a_1\omega^2 W = 0 \tag{39}$$

The boundary conditions can be simplified to the following:

At the fixed end ($x=0$):

$$W = \frac{dW}{dx} = 0 \tag{40}$$

At the free end ($x=L$):

$$\frac{d^2 W}{dx^2} = 0 \tag{41}$$

$$\frac{d^3 W}{dx^3} + \delta_2\frac{dW}{dx} = 0 \tag{42}$$

The general solution of Eq. (39) can be determined as

$$W = C_1\cos\mu_1 x + C_2\sin\mu_1 x + C_3\cosh\mu_2 x + C_4\sinh\mu_2 x \tag{43}$$

where

$$\mu_1 \equiv \sqrt{(\delta_2+\delta_5)/2}\,, \qquad \mu_2 \equiv \sqrt{(\delta_5-\delta_2)/2}$$

$$\delta_5 \equiv \sqrt{\delta_2^2 + 4\delta_4^2}$$

Again, the boundary conditions reduce to the form expressed in Eq. (35) and the characteristic equation is in the same form as Eq. (36). Expressions for coefficients k_{11}, k_{12}, k_{21}, and k_{22} are listed in Appendix A.

Free Vibration of a Sandwich Tapered Beam

For a tapered sandwich beam, the depths of the cord and facings vary linearly along the longitudinal axis of the beam. Their edge coordinates can be expressed as

$$h_1 = h_{1_o} - s_1 x \qquad h_2 = h_{2_o} - s_2 x \qquad h_3 = h_{3_o} - s_3 x \tag{44}$$

where $s_1 \equiv (h_{1_o} - h_{1_L})/L$, $s_2 \equiv (h_{2_o} - h_{2_L})/L$, and $s_3 \equiv (h_{3_o} - h_{3_L})/L$. The quantities h_{1_o}, h_{2_o}, h_{3_o} are the edge dimensions at the fixed end ($x=0$) of the beam, and h_{1_L}, h_{2_L}, h_{3_L} are the corresponding dimensions at the free end ($x=L$).

The coefficients of the governing equations and boundary conditions become functions of x, as follows:

$$a_1 = c_1 - b_1 x \qquad a_2 = c_2 - d_1 x + b_2 x^2$$

$$a_3 = c_3 - d_2 x + d_3 x^2 - b_3 x^3 \qquad a_4 = c_4 - b_4 x$$

$$a_5 = c_5 - d_4 x + b_5 x^2 \qquad a_6 = c_6 - d_5 x + d_6 x^2 - b_6 x^3$$

$$a_7 = c_7 - b_7 x \qquad \alpha_1 = d_5 - 2d_6 x + 3b_6 x^2$$

$$\alpha_2 = d_2 - 2d_3 x + 3b_3 x^2 \qquad \alpha_3 = 2b_5 x - d_4$$

$$\alpha_4 = 2b_2 x - d_1 \qquad \alpha_5 = 3b_6 x - d_6 \tag{45}$$

where the constants b, c, and d are defined in Appendix B.

Timoshenko Beams

The governing equations, Eqs. (1-3), can be simplified to the following for a nonsymmetrical tapered beam:

$$a_1\frac{\partial^2 u_o}{\partial t^2} - a_2\left(\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + a_5\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^3 w_o}{\partial x^3}\right)$$
$$+ \alpha_3\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) - a_4\frac{\partial^2 u_o}{\partial x^2} + b_4\frac{\partial u_o}{\partial x}$$
$$+ \Omega^2\left[a_2\left(\psi + \frac{\partial w_o}{\partial x}\right) - a_1 u_o\right] = 0 \tag{46}$$

$$a_3\left(\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^3 w_o}{\partial x \partial t^2}\right) + a_7\psi - a_2\frac{\partial^2 u_o}{\partial t^2} + a_5\frac{\partial^2 u_o}{\partial x^2} + \alpha_3\frac{\partial u_o}{\partial x}$$
$$- a_6\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^3 w_o}{\partial x^3}\right) + \alpha_1\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) + \Omega^2 a_2 u_o$$
$$- a_3\Omega^2\left(\psi + \frac{\partial w_o}{\partial x}\right) = 0 \tag{47}$$

$$a_1\frac{\partial^2 w_o}{\partial t^2} + a_2\frac{\partial^3 u_o}{\partial t^2 \partial x} + \alpha_4\frac{\partial^2 u_o}{\partial t^2} - a_3\left(\frac{\partial^4 w_o}{\partial x^2 \partial t^2} + \frac{\partial^3\psi}{\partial x \partial t^2}\right)$$
$$+ \alpha_2\left(\frac{\partial^3 w_o}{\partial x \partial t^2} + \frac{\partial^2\psi}{\partial t^2}\right) + a_6\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^4 w_o}{\partial x^4}\right)$$
$$- 2\alpha_1\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^3 w_o}{\partial x^3}\right) - 2\alpha_5\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) - a_5\frac{\partial^3 u_o}{\partial x^3}$$
$$- 2\alpha_3\frac{\partial^2 u_o}{\partial x^2} - 2b_5\frac{\partial u_o}{\partial x} + \Omega^2 a_3\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right)$$
$$- \alpha_2\Omega^2\left(\psi + \frac{\partial w_o}{\partial x}\right) - a_2\Omega^2\frac{\partial u_o}{\partial x} - \alpha_4\Omega^2 u_o = 0 \tag{48}$$

Combining Eq. (47) and Eq. (48), one obtains

$$\frac{\partial\psi}{\partial x} = \frac{b_7}{a_7}\psi - \frac{a_1}{a_7}\frac{\partial^2 w_o}{\partial t^2} \tag{49}$$

The displacement functions can be assumed as follows for harmonic vibration:

$$u_o = \mathrm{U}\sin(\omega t + \theta) \tag{50}$$

$$w_o = W\sin(\omega t + \theta) \tag{51}$$

$$\psi = \Phi\sin(\omega t + \theta) \tag{52}$$

where U is the angular frequencies and θ is the phase angle for all three modes. Substituting these functions into Eqs. (46–49), one can eliminate Φ and reduce the governing equations to

$$(a_5 D_1 + a_4 D_4)\frac{d^2\mathrm{U}}{dx^2} + (\alpha_3 D_1 - b_4 D_4)\frac{d\mathrm{U}}{dx}$$
$$+ (\Omega^2+\omega^2)(a_2 D_1 + a_1 D_4)\mathrm{U} - (a_6 D_1 + a_5 D_4)\frac{d^3 W}{dx^3}$$
$$+ (\alpha_1 D_1 - \alpha_3 D_4)\frac{d^2 W}{dx^2} + (D_1 D_5 - D_2 D_4)\frac{dW}{dx}$$
$$+ (D_1 D_6 - D_3 D_4)W = 0 \tag{53}$$

$$\left[2D_4(b_4 a_5 + \alpha_3 a_4) - a_5(a_5 D_7 - a_4 D_{12})\right]\frac{d^2\mathrm{U}}{dx^2}$$
$$- \left[a_1 a_5 D_4(\omega^2+\Omega^2) + a_4 D_4 D_8 + \alpha_3(a_5 D_7 - a_4 D_{12})\right]\frac{d\mathrm{U}}{dx}$$
$$+ \left[D_4(b_1 a_5 + \alpha_4 a_4)(\omega^2+\Omega^2) + a_2(a_4 D_{12} - a_5 D_7)(\omega^2+\Omega^2)\right]\mathrm{U}$$

$$+ D_4(a_5^2 - a_4a_6)\frac{d^4W}{dx^4} + \Big[2D_4(\alpha_3D_5 + \alpha_1a_4) + a_6(a_5D_7 - a_4D_{12})\Big]\frac{d^3W}{dx^3} + \Big[D_4(a_5D_{13} - a_4D_9) + \alpha_1(a_4D_{12} - a_5D_7)\Big]\frac{d^2W}{dx^2} + \Big[D_4(a_5D_{14} - a_4D_{10}) + D_5(a_4D_{12} - a_5D_7)\Big]\frac{dW}{dx} + \Big[D_4(a_5D_{15} - a_4D_{11}) - D_6(a_5D_7 - a_4D_{12})\Big]W = 0 \tag{54}$$

where the functions D are defined in Appendix B.

The boundary conditions at the fixed end ($x = 0$) are identical to those given in Eq. (25). In addition to Eq. (26), boundary conditions at the free end ($x = L$) include the following two relations, which can be reduced from Eqs. (6) and (7):

$$a_1\omega^2W + a_7\frac{d^2W}{dx^2} = 0 \tag{55}$$

$$-(\Omega^2 + \omega^2)a_2\mathsf{U} - a_5\frac{d^2\mathsf{U}}{dx^2} + a_6\frac{d^3W}{dx^3} + \Big[a_3(\omega^2 + \Omega^2) + \frac{a_6a_1}{a_7}\omega^2\Big] \times \frac{dW}{dx} + a_6\left(\frac{2b_7a_1}{a_7^2} - \frac{b_1}{a_7}\right)\omega^2W = 0 \tag{56}$$

Approximate techniques, such as the finite-difference method, should be used to solve the eigenvalue problem, since the coefficients of the governing equations and boundary conditions are not constant and consequently obtaining the exact form of the corresponding characteristic equation becomes practically infeasible.

For a symmetrically tapered beam, Eqs. (46–48) reduce to

$$a_1\frac{\partial^2u_o}{\partial t^2} - a_4\frac{\partial^2u_o}{\partial x^2} + b_4\frac{\partial u_o}{\partial x} - a_1\Omega^2u_o = 0 \tag{57}$$

$$a_3\left(\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^3w_o}{\partial x\partial t^2}\right) + a_7\psi - a_6\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^3w_o}{\partial x^3}\right) + \alpha_1\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2w_o}{\partial x^2}\right) - a_3\Omega^2\left(\psi + \frac{\partial w_o}{\partial x}\right) = 0 \tag{58}$$

$$a_1\frac{\partial^2w_o}{\partial t^2} - a_3\left(\frac{\partial^3\psi}{\partial x\partial t^2} + \frac{\partial^4w_o}{\partial x^2\partial t^2}\right) + \alpha_2\left(\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^3w_o}{\partial x\partial t^2}\right) + a_6\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^4w_o}{\partial x^4}\right) - 2\alpha_1\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^3w_o}{\partial x^3}\right) - 2\alpha_5\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2w_o}{\partial x^2}\right) + \Omega^2a_3\left(\frac{\partial\psi}{\partial x} + \frac{\partial^2w_o}{\partial x^2}\right) - \alpha_2\Omega^2\left(\psi + \frac{\partial w_o}{\partial x}\right) = 0 \tag{59}$$

Since the axial mode decouples from the other two modes, the natural frequencies of this mode may not be the same as those of the other modes in general. Hence, displacement functions should be assumed in the forms of Eqs. (13), (14), and (19). The governing equations then can be simplified to the following by eliminating ψ from Eqs. (55) and (56):

$$a_4\frac{d^2\mathsf{U}}{dx^2} - b_4\frac{d\mathsf{U}}{dx} + a_1(\omega_1^2 + \Omega^2)\mathsf{U} = 0 \tag{60}$$

$$a_6D_4\frac{d^4W}{dx^4} + (a_6D_7 - 2\alpha_1D_4)\frac{d^3W}{dx^3} + (D_4D_9 - \alpha_1D_7)\frac{d^2W}{dx^2} + (D_4D_{10} - D_5D_7)\frac{dW}{dx} + (D_4D_{11} - D_6D_7)W = 0 \tag{61}$$

The boundary conditions are identical to those for the nonsymmetrical case except that Eq. (56) can be reduced to:

$$a_6\frac{d^3W}{dx^3} + \Big[a_3(\omega^2 + \Omega^2) + \frac{a_1a_6}{a_7}\omega^2\Big]\frac{dW}{dx} + a_6\left(\frac{2a_1b_7}{a_7^2} - \frac{b_1}{a_7}\right)\omega^2W = 0 \tag{62}$$

Euler-Bernoulli Beams

The governing equations, Eqs. (8) and (9), reduce to the following for a nonsymmetrical tapered beam:

$$a_1\frac{\partial^2u_o}{\partial t^2} - a_4\frac{\partial^2u_o}{\partial x^2} + b_4\frac{\partial u_o}{\partial x} - a_1\Omega^2u_o - a_2\frac{\partial^3w_o}{\partial x\partial t^2} + a_5\frac{\partial^3w_o}{\partial x^3} + \alpha_3\frac{\partial^2w_o}{\partial x^2} + \Omega^2a_2\frac{\partial w_o}{\partial x} = 0 \tag{63}$$

$$a_1\frac{\partial^2w_o}{\partial t^2} - a_3\frac{\partial^4w_o}{\partial x^2\partial t^2} + \alpha_2\frac{\partial^3w_o}{\partial x\partial t^2} + a_6\frac{\partial^4w_o}{\partial x^4} - 2\alpha_1\frac{\partial^3w_o}{\partial x^3} + (\Omega^2a_3 - 2\alpha_5)\frac{\partial^2w_o}{\partial x^2} - \alpha_2\Omega^2\frac{\partial w_o}{\partial x} + a_2\frac{\partial^3u_o}{\partial x\partial t^2} + \alpha_4\frac{\partial^2u_o}{\partial t^2} - a_5\frac{\partial^3u_o}{\partial x^3} - 2\alpha_3\frac{\partial^2u_o}{\partial x^2} - (a_2\Omega^2 + 2b_5)\frac{\partial u_o}{\partial x} - \alpha_4\Omega^2u_o = 0 \tag{64}$$

The displacement functions expressed in Eqs. (50) and (51) can be utilized to simplify Eqs. (63) and (64) to

$$a_4\frac{d^2\mathsf{U}}{dx^2} - b_4\frac{d\mathsf{U}}{dx} + a_1(\omega^2 + \Omega^2)\mathsf{U} - a_5\frac{d^3W}{dx^3} - \alpha_3\frac{d^3W}{dx^3} - a_2(\Omega^2 + \omega^2)\frac{dW}{dx} = 0 \tag{65}$$

$$(a_4a_6 - a_5^2)\frac{d^4W}{dx^4} - 2(\alpha_1a_4 + \alpha_3a_5)\frac{d^3W}{dx^3} + \Big[(a_3a_4 - a_2a_5)(\Omega^2 + \omega^2) - 2(\alpha_5a_4 + b_5a_5)\Big]\frac{d^2W}{dx^2} - \Big[(\alpha_2a_4 + \alpha_4a_5)(\Omega^2 + \omega^2)\Big]\frac{dW}{dx} - a_1a_4\omega^2W - 2(\alpha_3a_4 + b_4a_5)\frac{d^2\mathsf{U}}{dx^2} + \Big[(a_1a_5 - a_2a_4)(\Omega^2 + \omega^2) - 2b_5a_4\Big]\frac{d\mathsf{U}}{dx} - (\alpha_4a_4 + b_1a_5)(\Omega^2 + \omega^2)\mathsf{U} = 0 \tag{66}$$

The boundary conditions can be deduced from Eqs. (10), (11), and (12) as

At the fixed end ($x = 0$):

$$\mathsf{U} = W = \frac{dW}{dx} = 0 \tag{67}$$

At the free end ($x = L$):

$$\frac{d\mathsf{U}}{dx} = \frac{d^2W}{dx^2} = 0 \tag{68}$$

$$a_6\frac{d^3W}{dx^3} + a_3(\Omega^2 + \omega^2)\frac{dW}{dx} - a_5\frac{d^2\mathsf{U}}{dx^2} - a_2(\Omega^2 + \omega^2)\mathsf{U} = 0 \tag{69}$$

For a symmetrically tapered beam, Eqs. (63) and (64) reduce to

$$a_1\frac{\partial^2u_o}{\partial t^2} - a_4\frac{\partial^2u_o}{\partial x^2} + b_4\frac{\partial u_o}{\partial x} - a_1\Omega^2u_o = 0 \tag{70}$$

Table 1 Natural frequencies of the longitudinal (ω_1) and the transverse (ω_2) vibrations of symmetric sandwich beams at various rotating speeds (Ω)

		Beam A (rectangular)			Beam C (tapered)		
		ω_1 (Hertz)	ω_2 (Hertz)		ω_1 (Hertz)	ω_2 (Hertz)	
Mode number	Ω (Hertz)	TBM & EBM	TBM	EBM	TBM & EBM	TBM	EBM
1	0	421.9	27.12	27.09	444.2	26.64	26.55
	50	419.0	26.94	26.91	441.3	26.54	26.44
	100	409.9	26.41	26.37	432.8	26.22	26.12
	200	371.5	24.11	24.07	396.6	24.90	24.80
	300	296.7	19.53	19.47	327.5	22.50	22.40
	400	134.3	9.18	9.14	193.1	18.54	18.43
	420	40.4	3.38	3.36	144.5	17.46	17.36
	500	—	—	—	—	11.29	11.21
	540	—	—	—	—	4.96	4.93
	600	—	—	—	—	—	—
2	0	1266	165.2	167.8	1274	139.4	140.3
	50	1265	165.0	167.7	1273	139.2	140.2
	100	1262	164.5	167.1	1270	138.9	139.8
	200	1250	162.2	164.7	1258	137.5	138.4
	300	1230	158.4	160.7	1238	135.1	135.9
	400	1201	152.8	154.9	1210	131.7	132.4
	500	1163	145.4	147.1	1172	127.2	127.8
	700	1115	135.6	137.1	1124	121.5	121.9
3	0	2110	439.6	461.8	2114	358.8	369.2
	50	2109	439.5	461.6	2113	358.7	369.1
	100	2107	439.1	461.1	2112	358.4	368.8
	200	2100	437.4	459.2	2105	357.2	367.4
	300	2088	434.6	455.9	2093	355.2	365.2
	400	2071	430.1	451.3	2076	352.4	362.1
	500	2050	425.5	445.3	2054	348.7	358.0
	600	2023	419.1	437.9	2027	344.2	352.9

$$a_1\frac{\partial^2 w_o}{\partial t^2}-a_3\frac{\partial^4 w_o}{\partial x^2\partial t^2}+\alpha_2\frac{\partial^3 w_o}{\partial x\partial t^2}+a_6\frac{\partial^4 w_o}{\partial x^4}-2\alpha_1\frac{\partial^3 w_o}{\partial x^3}+(\Omega^2 a_3-2\alpha_5)\frac{\partial^2 w_o}{\partial x^2}-\alpha_2\Omega^2\frac{\partial w_o}{\partial x}=0 \quad (71)$$

Since governing equations become decoupled, the displacement functions should be assumed in the forms of Eqs. (13) and (14). Equations (70) and (71), therefore, become

$$a_4\frac{d^2U}{dx^2}-b_4\frac{dU}{dx}+a_1(\Omega^2+\omega_1^2)U=0 \quad (72)$$

$$a_6\frac{d^4W}{dx^4}-2\alpha_1\frac{d^3W}{dx^3}+\left[a_3(\Omega^2+\omega_2^2)-2\alpha_5\right]\frac{d^2W}{dx^2}-\alpha_2(\Omega^2+\omega_2^2)\frac{dW}{dx}-a_1\omega_2^2W=0 \quad (73)$$

The boundary condition at $x=L$ expressed in Eq. (69) becomes

$$a_6\frac{d^3W}{dx^3}+a_3(\Omega^2+\omega_2^2)\frac{dW}{dx}=0 \quad (74)$$

Since the coefficients a and α are functions of x, numerical techniques should be employed to solve these eigenvalue problems.

Numerical Examples

Four cantilever sandwich rotating beams are considered for numerical computations: a symmetric rectangular beam (beam A), a nonsymmetric rectangular beam (beam B), a symmetrically tapered beam (beam C), and a nonsymmetrically tapered beam (beam D). Aluminum core with steel facings are modeled for the sandwich construction. Hence, the material properties are selected, as follows:

$$E_1=6.89\times10^{10}\text{ Pa }(10\times10^6\text{ psi})$$

$$E_2=E_3=2.07\times10^{11}\text{ Pa }(30\times10^6\text{ psi})$$

$$G_1=2.62\times10^{10}\text{ Pa }(3.80\times10^6\text{ psi})$$

$$G_2=G_3=8.00\times10^{10}\text{ Pa }(11.5\times10^6\text{ psi})$$

$$\rho_1=2643\text{ kg/m}^3\ (165\text{ lbm/ft}^3)$$

$$\rho_2=\rho_3=7770\text{ kg/m}^3\ (485\text{ lbm/ft}^3)$$

The shear correction factors are chosen to be 0.850 for both materials by following Cowper's formulation[23] for rectangular cross sections. The beam lengths of these beams are chosen to be 3.05 m (10 ft). The half-thickness of the core (h_1) is selected as 0.0762 m (3 in.) for rectangular beams (beams A and B). A uniform thickness of 0.0508 m (2 in.) is selected for both facings of beam A, whereas the thickness of the bottom facing of beam B is changed to 0.0762 m (3 in.) to model its nonsymmetric behavior. The core thickness of beams C and D are tapered from 0.0762 m (3 in.) at the fixed end to 0.0254 m (1 in.) at the free end. Both facings of these beams have uniform thickness. Beam C has a symmetric geometry with a 0.0508 m (2 in.) facing thickness on both sides, whereas the thickness of the bottom facing of beam D is changed to 0.0762 m (3 in.) to model its nonsymmetric characteristic.

Governing equations for both Timoshenko's beam model (TBM) and Euler-Bernoulli's model (EBM) were solved by the finite-difference method, except those of beam A. Nonsymmetric beams (beams B and D) and the symmetrically tapered beam (beam C) were divided into 25 and 50 segments, respectivley. The eigenvalue problems were solved by evaluating determinants with an order of 50×50.

Table 1 shows the natural frequencies of the first three modes for both the transverse and the longitudinal vibrations of the symmetric beams (beams A and C). Results for beam A were obtained from the solutions listed in Appendix A. The values of natural frequencies for both the transverse and the extensional modes decrease with increasing rotating speed. The fundamental frequencies disappear at a high speed of rotation. Table 2 shows the natural frequencies of the first three modes for nonsymmetric beams (beams B and D). The

Table 2 Natural frequencies in Hertz for coupled vibrations of nonsymmetric sandwich beams at various rotating speeds (Ω)

Mode number	Ω (Hertz)	Beam B (rectangular)		Beam D (tapered)	
		TBM	EBM	TBM	EBM
	0	29.30	29.26	28.54	28.50
	50	29.11	29.07	28.42	28.39
	100	28.53	28.49	28.07	28.03
	200	36.06	26.00	26.59	26.54
1	300	21.10	21.08	23.87	23.81
	400	9.92	9.86	19.28	19.21
	420	3.62	3.60	17.97	17.91
	500	—	—	10.65	10.60
	530	—	—	4.51	4.49
	600	—	—	—	—
	0	176.8	180.0	150.7	152.4
	50	176.6	179.8	150.6	152.3
	100	176.0	179.2	150.2	151.9
	200	173.6	176.6	148.7	150.3
2	300	169.5	172.3	146.0	147.5
	400	163.6	166.1	142.3	143.6
	500	155.6	157.7	137.3	138.4
	600	145.1	146.9	130.8	131.7
	0	464.1	490.4	386.0	400.2
	50	463.9	490.2	385.9	400.1
	100	463.5	489.7	385.6	399.7
3	200	461.8	487.6	384.3	398.2
	300	458.8	484.1	382.1	395.8
	400	454.7	479.2	379.0	392.3
	500	449.3	472.8	375.1	387.8
	600	442.5	464.9	370.1	382.2

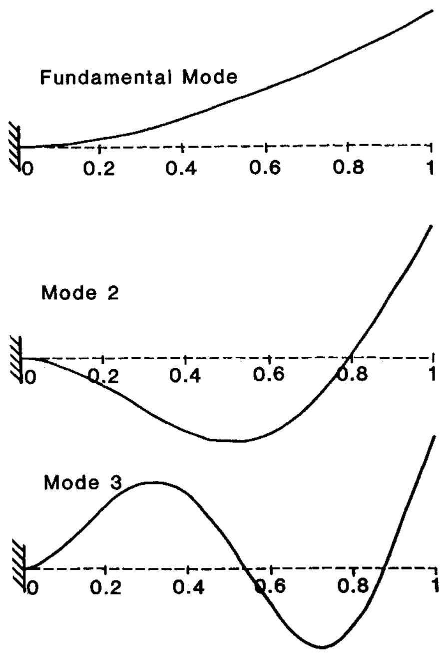

Fig. 2 Modal shapes for the transverse vibration of a rotating sandwich cantilever beam.

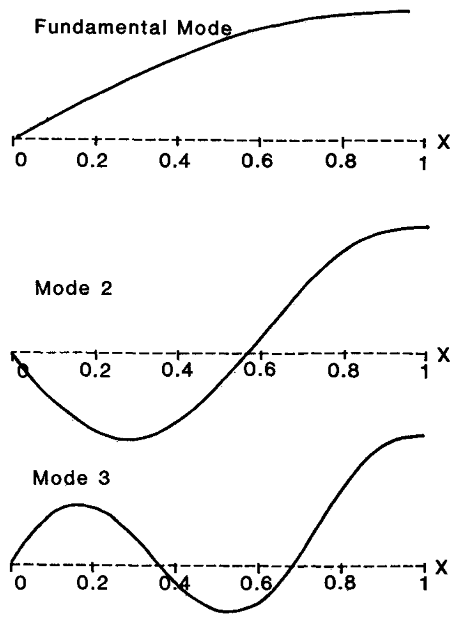

Fig. 3 Modal shapes for the longitudinal vibration of a rotating sandwich cantilever beam.

transverse and the extensional oscillations are coupled into one mode with a single frequency for these cases. Their values vary with the rotating speed in the same way as those of symmetric beams. For most cases, the EBM predicted higher frequencies than those calculated by TBM for higher modes; however, the opposite trend can also be observed for fundamental modes. The effect of beam geometry indicated by these results is inconclusive, also. However, for most cases, rectangular beams have higher frequencies for transverse or coupled vibrations and lower frequencies for longitudinal vibrations than their tapered counterparts. Typical modal shapes of the transverse and the longitudinal vibrations of a nonsymmetric sandwich beam are shown in Figs. 2 and 3, respectively. Those for symmetric beams do not differ qualitatively from those shown in these figures. Rotational speed also has little effect on the qualitative characteristics of these modal shapes, except that lower frequency modes vanish at an extremely high rotating speed.

The natural frequencies of a nonrotating homogeneous beam can be determined by treating it as a special case of a sandwich beam. For a nonrotating rectangular homogeneous beam, Eq. (31) reduces to the expression of natural frequencies for longitudinal vibration of a cantilever beam, given by Den Hartog.[24] For a steel beam with the same geometry as beam A, the frequencies of the first three modes can be determined as 423.3 Hz, 1270 Hz, and 2116 Hz, which are slightly higher

Table 3 Comparison of the first three natural frequencies in Hertz for transverse vibration of a homogeneous nonrotating steel cantilever beam (length = 10 ft, depth = 10 in.) calculated by using the present models and the classical Euler-Bernoulli's model (Ref. 24)

	Present models		
Mode	TBM	EBM	Classical EBM
1	22.78	22.77	22.82
2	140.0	141.6	142.6
3	377.5	391.4	400.1

than those of beam A. This is reasonable because beam A is less stiff than the homogeneous steel beam. Comparison of the frequencies for transverse vibrations of this homogeneous steel beam calculated by the present models with those calculated by the formula given by Den Hartog[24] is shown in Table 3. Both EBM and TBM predict slightly lower vlaues than those calculated by the classical method. These results indicate that the formulation, by including the neutral axis variation, predicts lower frequencies than those calculated by the classical method. The effect of this neutral-axis variation is not very significant for low-frequency modes due to the homogeneous and nonrotating characteristics of the beam.

Conclusions

The governing equations for the free vibration of a rotating sandwich beam have been solved numerically and reasonable results have been obtained. These equations include the effect of the variation of the neutral axis position along the beam. This effect can be very significant for a rotating sandwich beam, due to both the centrifugal force and a nonsymmetric distribution of property or geometry.

The natural frequencies of a rotating beam have been shown to vary with the rotating speed. In addition, at a high speed, the lower frequency modes or the fundamental mode can vanish. This indicates that the dynamic behavior of a rotating sandwich beam can differ drastically from that of a nonrotating sandwich beam.

Appendix A: Coefficients of the Characteristic Equation for the Transverse Response of a Symmetric Rectangular Sandwich Beam

Coefficients of the characteristic equation, Eq. (36), are defined as shown below:

Timoshenko Beams

Case 1: $a_3(\omega_2^2+\Omega^2)>a_7$

$$\kappa_{11}\equiv(\beta_2^2-\delta_1)\cos\beta_2L+(\delta_1-\beta_1^2)\cos\beta_1L$$

$$\kappa_{12}\equiv(\delta_1\beta_1^2)\sin\beta_1L+\left(\beta_1\beta_2-\frac{\delta_1\beta_1}{\beta_2}\right)\sin\beta_2L$$

$$\kappa_{21}\equiv\beta_1(\beta_1^2-\delta_1-\delta_2)\sin\beta_1L+\beta_2(\delta_1+\delta_2-\beta_2^2)\sin\beta_2L$$

$$\kappa_{22}\equiv\beta_1(\delta_1+\delta_2-\beta_1^2)\cos\beta_1L+\beta_1(\beta_2^2-\delta_1-\delta_2)\cos\beta_2L$$

Case 2: $a_3(\omega_2^2+\Omega^2)<a_7$

$$\kappa_{11}\equiv(\delta_1-\beta_1^2)\cos\beta_1L-(\delta_1+\beta_3^2)\cosh\beta_3L$$

$$\kappa_{12}\equiv(\delta_1-\beta_1^2)\sin\beta_1L-\frac{\beta_1}{\beta_3}(\delta_1+\beta_3^2)\sinh\beta_3L$$

$$\kappa_{21}\equiv\beta_1(\beta_1^2-\delta_1-\delta_2)\sin\beta_1L-\beta_3(\delta_1+\delta_2+\beta_3^2)\sinh\beta_3L$$

$$\kappa_{22}\equiv\beta_1(\delta_1+\delta_2-\beta_1^2)\cos\beta_1L-\beta_1(\delta_1+\delta_2+\beta_3^2)\cosh\beta_3L$$

Case 3: $a_3(\omega_2^2+\Omega^2)=a_7$

$$\kappa_{11}\equiv(\delta_1-\beta_1^2)\cos\beta_1L-\delta_1$$

$$\kappa_{12}\equiv(\delta_1-\beta_1^2)\sin\beta_1L-\beta_1L$$

$$\kappa_{21}\equiv\beta_1(\beta_1^2-\delta_1-\delta_2)\sin\beta_1L$$

$$\kappa_{22}\equiv\beta_1(\delta_1+\delta_2-\beta_1^2)\cos\beta_1L-\beta_1(\delta_1+\delta_2)$$

Euler-Bernoulli Beams

$$\kappa_{11}\equiv\mu_1^2\cos\mu_1L+\mu_2^2\cosh\mu_2L$$

$$\kappa_{12}\equiv\mu_1^2\sin\mu_1L+\mu_1\mu_2\sinh\mu_2L$$

$$\kappa_{21}\equiv\mu_1(\mu_1^2-\delta_2)\sin\mu_1L-\mu_2(\mu_2^2+\delta_2)\sinh\mu_2L$$

$$\kappa_{22}\equiv\mu_1(\delta_2-\mu_1^2)\cos\mu_1L-\mu_1(\mu_2^2+\delta_2)\cosh\mu_2L$$

Appendix B: Definitions of Coefficients of the Governing Equations for a Rotating Tapered Sandwich Beam

$$c_1\equiv2\rho_1h_{1_o}+\rho_2(h_{2_o}-h_{1_o})+\rho_3(h_{3_o}-h_{1_o})$$

$$c_2\equiv\left[\rho_2(h_{1_o}^2-h_{2_o}^2)+\rho_3(h_{3_o}^2-h_{1_o}^2)\right]/2$$

$$c_3\equiv\left[2\rho_1h_{1_o}^3+\rho_2(h_{2_o}^3-h_{1_o}^3)+\rho_3(h_{3_o}^3-h_{1_o}^3)\right]/3$$

$$c_4\equiv2E_1h_{1_o}+E_2(h_{2_o}-h_{1_o})+E_3(h_{3_o}-h_{1_o})$$

$$c_5\equiv\left[E_2(h_{1_o}^2-h_{2_o}^2)+E_3(h_{3_o}^2-h_{1_o}^2)\right]/2$$

$$c_6\equiv\left[2E_1h_{1_o}^3+E_2(h_{2_o}^3-h_{1_o}^3)+E_3(h_{3_o}^3-h_{1_o}^3)\right]/3$$

$$c_7\equiv2k_1G_1h_{1_o}+k_2G_2(h_{2_o}-h_{1_o})+k_3G_3(h_{3_o}-h_{1_o})$$

$$b_1\equiv2\rho_1s_1+\rho_2(s_2-s_1)+\rho_3(s_3-s_1)$$

$$b_2\equiv\left[\rho_2(s_1^2-s_2^2)+\rho_3(s_3^2-s_1^2)\right]/2$$

$$b_3\equiv\left[2\rho_1s_1^3+\rho_2(s_2^3-s_1^3)+\rho_3(s_3^3-s_1^3)\right]/3$$

$$b_4\equiv2E_1s_1+E_2(s_2-s_1)+E_3(s_3-s_1)$$

$$b_5\equiv\left[E_2(s_1^2-s_2^2)+E_3(s_3^2-s_1^2)\right]/2$$

$$b_6\equiv\left[2E_1s_1^3+E_2(s_2^3-s_1^3)+E_3(s_3^3-s_1^3)\right]/3$$

$$b_7\equiv2k_1G_1s_1+k_2G_2(s_2-s_1)+k_3G_3(s_3-s_1)$$

$$d_1\equiv\rho_2(s_1h_{1_o}-s_2h_{2_o})+\rho_3(s_3h_{3_o}-s_1h_{1_o})$$

$$d_2\equiv2\rho_1s_1h_{1_o}^2+\rho_2(s_2h_{2_o}^2-s_1h_{1_o}^2)+\rho_3(s_3h_{3_o}^2-s_1h_{1_o}^2)$$

$$d_3\equiv2\rho_1s_1^2h_{1_o}+\rho_2(s_2^2h_{2_o}-s_1^2h_{1_o})+\rho_3(s_3^2h_{3_o}-s_1^2h_{1_o})$$

$$d_4\equiv E_2(s_1h_{1_o}-s_2h_{2_o})+E_3(s_3h_{3_o}-s_1h_{1_o})$$

$$d_5\equiv2E_1s_1h_{1_o}^2+E_2(s_2h_{2_o}^2-s_1h_{1_o}^2)+E_3(s_3h_{3_o}^2-s_1h_{1_o}^2)$$

$$d_6\equiv2E_1s_1^2h_{1_o}+E_2(s_2^2h_{2_o}-s_1^2h_{1_o})+E_3(s_3^2h_{3_o}-s_1^2h_{1_o})$$

$$D_1\equiv a_2(\Omega^2+\omega^2)+\frac{2a_5b_7^2}{a_7^2}+\frac{\alpha_3b_7}{a_7}$$

$$D_2\equiv a_2(\Omega^2+\omega^2)+\frac{a_1a_5\omega^2}{a_7}$$

$$D_3\equiv\frac{\omega^2}{a_7}\left(\frac{2a_1a_5b_7}{a_7}+a_1\alpha_3-a_5b_1\right)$$

$$D_4 \equiv a_7 + \frac{b_7\alpha_1}{a_7} - \frac{2a_6b_7}{a_7^2} - a_3(\Omega^2+\omega^2)$$

$$D_5 \equiv -a_3(\Omega^2+\omega^2) - \frac{a_1a_6\omega^2}{a_7}$$

$$D_6 \equiv \frac{\omega^2}{a_7}\left(a_6b_1+\alpha_1a_1-\frac{2a_1a_6b_7}{a_7}\right)$$

$$D_7 \equiv \frac{b_7a_3(\Omega^2+\omega^2)}{a_7} + \frac{6a_6b_7^3}{a_7^3} - \frac{4\alpha_1b_7^2}{a_7^2} - \frac{2b_7\alpha_5}{a_7} - \alpha_2(\Omega^2+\omega^2)$$

$$D_8 \equiv 2b_5 + a_2(\Omega^2+\omega^2)$$

$$D_9 \equiv a_3(\Omega^2+\omega^2) + \frac{a_1a_6\omega^2}{a_7}$$

$$D_{10} \equiv \frac{3a_1a_6b_7\omega^2}{a_7^2} - \frac{2a_1\alpha_1\omega^2}{a_7} - \frac{2a_6b_1\omega^2}{a_7} - \alpha_2(\Omega^2+\omega^2)$$

$$D_{11} \equiv \omega^2\left[\frac{6a_1a_6b_7}{a_7^3} - \frac{b_7}{a_7^2}(3b_1a_6+4a_1\alpha_1) + \frac{2}{a_7}(b_1\alpha_1-a_1\alpha_5) + \frac{a_1a_3}{a_7}(\Omega^2+\omega^2) - a_1\right]$$

$$D_{12} \equiv (\Omega^2+\omega^2)\left(\alpha_4+\frac{a_2b_7}{a_7}\right) + \frac{2b_7}{a_7}\left(b_5+\frac{2b_7\alpha_3}{a_7}+\frac{3b_7^2a_5}{a_7^2}\right)$$

$$D_{13} \equiv 2b_5 + D_2$$

$$D_{14} \equiv \alpha_4(\Omega^2+\omega^2) + \frac{\omega^2}{a_7}\left(2a_1\alpha_3-2b_1a_5+\frac{3a_1a_5b_7}{a_7}\right)$$

$$D_{15} \equiv \frac{\omega^2}{a_7}\left[\frac{b_7}{a_7}\left(\frac{6a_1a_5b_7}{a_7}+4a_1\alpha_3-3a_5b_1\right) + a_1a_2(\Omega^2+\omega^2) + 2a_1b_5-2b_1\alpha_3\right]$$

References

[1]Lo, H. and Renbarger, J. L., "Bending Vibrations of a Rotating Beam," *Proceedings of the First U.S. National Congress of Applied Mechanics,* Illinois Inst. of Technology, Chicago, IL, 1951, pp. 75–79.

[2]Boyce, W. E., Di Prima, R. C., and Handelman, G. H., "Vibrations of Rotating Beams of Constant Section," *Proceedings of the Second U.S. National Congress of Applied Mechanics,* American Society of Mechanical Engineers, New York, Univ. of Michigan, Ann Arbor, MI, 1954, pp. 165–172.

[3]Handelman, G. H., Boyce, W. E., and Cohen, H., "Vibrations of a Uniform Rotating Beam with Tip Mass," *Proceedings of the Third U.S. National Congress of Applied Mechanics,* American Society of Mechanical Engineers, New York, 1958, pp. 175–180.

[4]Boyce, W. E. and Handelman, G. H., "Vibrations of Rotating Beam with Tip Mass," *Zeitschrift für angewandte Mathematik und Physik,* Vol. 12, No. 5, 1961, pp. 369–392.

[5]Jones, L. H., "The Transverse Vibration of a Rotating Beam with Tip Mass; The Method of Integral Equations," *Quarterly of Applied Mathematics,* Vol. 33, Oct. 1975, pp. 193–203.

[6]Hoa, S. V., "Vibration of a Rotating Beam with Tip Mass," *Journal of Sound and Vibration,* Vol. 67, March 1979, pp. 369–375.

[7]Tomar, J. S. and Jain, R., "Thermal Effect on the Frequencies of a Rotating Wedge-Shaped Beam," *AIAA Journal,* Vol. 22, June 1984, pp. 848–850.

[8]Tomar, J. S. and Jain, R., "Thermal Effect on Frequencies of Coupled Vibrations of Pretwisted Rotating Beams," *AIAA Journal,* Vol. 23, Aug. 1985, pp. 1293–1296.

[9]Laurensen, R. M., "Influence of Mass Representation on the Equation of Motion for Rotating Structures," *AIAA Journal,* Vol. 23, Dec. 1985, pp. 1995–1998.

[10]Wright, A. D., Smith, C. E., Thresher, R. W., and Wang, J. L. C., "Vibration Modes of Centrifugally Stiffened Beams," *Transactions of ASME, Journal of Applied Mechanics,* Vol. 49, March 1982, pp. 197–202.

[11]Bruch, J. C. and Mitchell, T. P., "Vibrations of a Mass-Loaded Clamped-Free Timoshenko Beam," *Journal of Sound and Vibration,* Vol. 114, Feb. 1987, pp. 341–345.

[12]Bert, C. W., "Analysis of Plates," *Composite Materials, Structural Design and Analysis, Part I,* Vol. 7, edited by C. C. Chamis, Academic, New York, 1974, pp. 194–198.

[13]Kobayashi, S., "On Vibrations of Sandwich Beams," *Proceedings of the Fourth Japan National Congress of Applied Mechanics,* American Society of Mechanical Engineers, New York, 1954, pp. 369–372.

[14]Krajcinovic, D., "Vibrations of Laminated Beams," AIAA Paper 72-399 (AIAA/ASME/SAE 13th Structures, Structural Dynamics, and Materials Conference, San Antonio, TX, April 10–12, 1972).

[15]Kimel, W. R., Raville, M. E., Kirmser, P. G., and Patel, M. P., "Natural Frequencies of Vibration of Simply Supported Sandwich Beams," *Proceedings of the Fourth Midwestern Conference on Solid Mechanics,* University of Texas Press, Austin, TX, 1959, pp. 441–456.

[16]Raville, M. E., Ueng, E. -S., and Lei, M. -M., "Natural Frequencies of Vibration of Fixed-Fixed Sandwich Beams," *Transactions of ASME, Journal of Applied Mechanics,* Vol. 28, Sept. 1961, pp. 376–381.

[17]James, W. L., "Calculation of Vibration Damping in Sandwich Construction from Damping Properties of the Core and Facings," Forest Products Laboratory, Madison, WI, Rept. No. 1888, Dec. 1962.

[18]Glaser, A. R., "The Vibration of Sandwich Beams," *Developments in Mechanics,* Vol. 1 (Proceedings of the Seventh Midwestern Mechanics Conference), Plenum Press, New York, 1961, pp. 228–238.

[19]Clary, R. R. and Leadbetter, S. A., "An Analytical and Experimental Investigation of the Natural Frequencies of Uniform Rectangular-Cross-Section Free-Free Sandwich Beams," NASA TN D-1967, Oct. 1963.

[20]Bert, C. W., Wilkins, D. J., Jr., and Crisman, W. C., "Damping in Sandwich Beams with Shear-Flexible Cores," *Transactions of ASME, Journal of Engineering for Industry,* Vol. 89, Nov. 1967, pp. 662–670.

[21]Ko, C. L., "Flexural Behavior of a Rotating Sandwich Tapered Beam," *AIAA Journal,* Vol. 27, March 1989, pp. 359–369.

[22]Knoell, A. C. and Robinson, E. Y., "Analysis of Truss, Beam, Frame, and Membrane Components," *Composite Materials, Structural Design and Analysis, Part I,* Vol. 7, edited by C. C. Chamis, Academic, New York, 1974, pp. 137–139.

[23]Cowper, G. R., "The Shear Coefficient in Timoshenko's Beam Theory," *Transactions of ASME, Journal of Applied Mechanics,* Vol. 33, June 1966, pp. 335–340.

[24]Den Hartog, J. P., *Mechanical Vibrations,* McGraw-Hill, New York, 4th ed., 1956, pp. 431–452.